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Journal of Geometry and Physics 45 (2003) 54–74

JOURNAL OF
GEOMETRY AND
PHYSICS

www.elsevier.com/locate/jgp

The (fully) Nonlinear Maxwell Theory delineated[☆]

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Received 28 March 2002

Abstract

The goal of this paper is to sketch a broader outline of the mathematical structures present in the Nonlinear Maxwell Theory in continuation of work previously presented in [J. Reine Angew. Math. 514 (1999) 1; Phys. Lett. A 228 (1997) 347; A. Sowa, Commun. Math. Phys., cond-mat/9904204, in press]. In particular, I display new types of both dynamic and static solutions of the Nonlinear Maxwell Equations (NM). I point out how the resulting theory ties to the Quantum Mechanics of Correlated Electrons inasmuch as it provides a mesoscopic description of phenomena like non-resistive charge transport, static magnetic flux tubes, and charge stripes in a way consistent with both the phenomenology and the microscopic principles. In addition, I point at a bunch of geometric structures intrinsic for the theory. On one hand, the presence of these structures indicates that the equations at hand can be used as ‘probing tools’ for purely geometric exploration of low-dimensional manifolds. On the other hand, global aspects of these structures are in my view prerequisite to incorporating (quantum) informational features of Correlated Electron Systems within the framework of the Nonlinear Maxwell Theory.

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MSC: 83C99

Subj. Class.: Classical field theory

Keywords: Nonlinear Maxwell Theory

1. Introduction

The general goal of this paper is to examine broader ramifications of the Nonlinear Maxwell Equations (NM) as introduced by me in 1992/1993 and further developed in [11–13]. To this end, I first point out that the theory is considerably richer than that of the

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classical linear Electromagnetism. In particular, I describe here several distinct types of both static and dynamic solutions on a spacetime of the form $M^3 \times R$. On the technical side, I have essentially avoided heavier analysis as the solutions are either obtained by means of elementary calculation, or are otherwise based on deeper analytic work described in [13]. One should be aware that the possibilities opening in consequence of the introduction of these new structures have not been fully exploited in this paper, thus postponing many potential developments into the future.

More precisely, in the ‘dynamic’ part of the paper I display a solution in the form of a charge-carrying electromagnetic wave. It is a soliton type wave that transports charge with constant speed and without resistance. In addition, one notes existence of a specific to dimension four nonlinear Fourier type transform—an interesting structure whose role within the theory is twofold. On one hand, it can be used to find and analyze new solutions of the Nonlinear Maxwell Equations. On the other hand, the transform defines an exotic duality—a (quadratic) generalization of the (linear) Hodge-duality. Consequences of this new duality for the four-geometry will be exploited in the future.

The second set of results in this paper is focused around the question of existence and properties of static solutions. To this end, I first examine the situation on the Euclidean three-space. In particular, one takes note of the occurrence of global structures in the form of magnetic flux tubes as well as the so-called charge stripes. It is interesting from the point of view of geometry that these objects exist in general only on three-manifolds whose fundamental group is not finite. This is tied to the geometric fact that the nonlinear gauge theory at hand induces an additional structure on M^3 —namely a taut codimension one foliation. These global aspects of static solutions prompt an assumption of topological point of view. Accordingly, I sketch the possibility of constructing ‘nonlinear cohomology’ that would account for a sort of ‘flux tube’ invariant of a three-manifold. The discussion here is based on two particular examples that I feel provide an optimal illustration of the underlying concept.

The Nonlinear Maxwell Equations, cf. (1)–(3) below, involve a vector potential that encodes the electric and magnetic fields in the usual way as well as an additional scalar f . The function f contains information, extractable in a certain simple canonical way, about the local value of the *filling factor* (also known as the filling fraction). (The filling factor is defined as the number of quanta of the magnetic field per electron charge in the first Landau level. It is then natural and effective to think of the electrons as forming in conjunction with the corresponding magnetic flux quanta composite particles—either bosons or fermions, or Laughlin particles depending on the actual value of the filling factor.) It is thus postulated that the filling fraction—typically an input of a microscopic theory that is always assumed constant microlocally—is allowed to slowly vary in the coarser scale. In fact, it was shown in [13] that NM predict occurrence of phase changes that lead to formation of vortices in f , and a fortiori in the magnetic field. This picture conforms with the well known analogy between the Quantum Hall Effects and the High- T_c Superconductivity. An inquisitive reader might now point at the following seeming conundrum. The physical interpretation of f as a filling factor requires the presence of two-dimensional geometric structures that endow us with a possibility of including the lowest Landau level in the basic dictionary. Thus, it may appear a priori puzzling, how we are going to retain this interpretation of f in 3 or 4 spatial dimensions? The answer is provided by the intrinsic structure of the NM themselves.

On one hand, it is shown below that the filling factor variable may be completely factored out of the equations when viewed in the complete four-dimensions of spacetime. Needless to say, if one attempted to analyze such f -free form in two-dimensions the f variable would reemerge without change as it is there encoded in the magnetic field $B = b/f$, $b = \text{const}$. On the other hand, one notes that a remnant or a generalization of the filling factor interpretation carries over to three-dimensions. Namely, the NM in three-dimensions imply existence of a codimension one foliation of the three-space associated with the static solutions. Moreover, one notes here that the NM do not a priori introduce any restrictions as to the type of the resulting foliation—in fact any regular foliation and even foliations with singularities introduced by degenerating leaves are admitted by the equations. However, as already mentioned above the existence of solutions of a special type, namely the flux tube type, implies geometrical restrictions on the foliation and topological restrictions on the three-manifold. Indeed, in this case foliation must be taut. It also seems reasonable to expect that the composite particle interpretation remains valid in this setting and the number of participating electrons in each leaf is again determined by f , virtually leading to the notion of an *effective* Landau level.

In the last words of this section I would like to admit that, the subject matter at hand being both new and inherently interdisciplinary as well as by way of my own background and limitations, it is not always easy to pick the optimal terminology. Realizing I will unavoidably fail to satisfy in this respect one group of readers or another, I can only ask the readership to be as tolerant as they can afford and hope that in the end substance will triumph over form.

2. Nonlinear Maxwell Equations in spacetime

In what follows, in order to get around rather tedious algebra while not compromising our understanding of what is essentially involved, I present a shortcut style exposition of the necessary calculations. I believe that readers who are well familiar with differential geometry will find it easy to reinterpret this calculation in its natural invariant setting, while those who are less familiar with the abstract setting may in fact appreciate its absence here.

Consider the following system of equations—the Nonlinear Maxwell Equations in the form in which they have appeared in my previous papers

$$dF_A = 0, \tag{1}$$

$$\delta(fF_A) = 0, \tag{2}$$

$$\square f + a|F_A|^2 f = \nu f, \tag{3}$$

where f is the real valued function and A is the electromagnetic vector potential, so that the corresponding electromagnetic field is $F_A = dA$. Here, $a > 0$ is a physical constant with unit $[\text{T}^{-2} \text{m}^{-2}]$; I will not discuss the precise physical interpretation of a^2 in this paper. Further, d is the exterior derivative and $\delta = \star d \star$ its adjoint. Here, it is assumed that the Hodge-star \star and the D'Alembertian \square are induced by the Lorentzian metric tensor on a spacetime of $3 + 1$ -dimensions. Let me point out that assuming $f = \text{const}$ and dragging it

to zero one recovers the classical Maxwell equations. In this sense, all phenomena of the classical Electromagnetism are included in the present model.

Now the goal is to better understand the essential ingredients of the NM in terms of the classical field variables. To this end, let us say the spacetime is in fact the flat Minkowski space (with the speed of light 1) so that in particular one can identify coefficients of the electric field \vec{E} and the magnetic field \vec{B} with the coefficients of the curvature tensor F_A by the formula

$$F_A = B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt. \tag{4}$$

For the sake of our discussion below it is good to keep in mind the well known fact that the components of F_A are not Lorentz invariant. This property leads one to the derivation of the Lorentz force, so that the latter one is logically independent of the particular form of a gauge theory formulated in terms of F_A . In other words, the Lorentz force remains unchanged and valid as one attempts to modify the field equations. With this understood, let us continue the discussion of Eqs. (1)–(3).

It would be rather straightforward to rewrite Eqs. (1)–(3) in the anticipated Maxwellian form by following the usual procedures for translating (2) into the Ampere and Gauss laws after having replaced \vec{E} by $f\vec{E}$ and \vec{B} by $f\vec{B}$. In fact, this would lead to an ad hoc interpretation of f as a material constant—a route taken in our older, one might say *naive*, paper [11]. However, this form of the system offers little insight as to the more essential implications of the NM, and one needs to find a less obvious reformulation.

One notes that Eq. (2) may be equivalently written in the form

$$f\delta F_A = F_A(\nabla_{3+1} f, \cdot), \tag{5}$$

where ∇_{3+1} stands for the gradient in spacetime. For a reason that will become clear later, one identifies F_A with a skew-symmetric matrix in a standard way

$$F = \begin{pmatrix} 0 & -B_3 & B_2 & -E_1 \\ B_3 & 0 & -B_1 & -E_2 \\ -B_2 & B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix}.$$

It is important to note that by the miraculous property of skew-symmetric matrices in four-dimensions, $\det F = \vec{E} \cdot \vec{B}$ and

$$F^{-1} = \frac{1}{\vec{E} \cdot \vec{B}} \begin{pmatrix} 0 & E_3 & -E_2 & B_1 \\ -E_3 & 0 & E_1 & B_2 \\ E_2 & -E_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{pmatrix} = \frac{1}{\vec{E} \cdot \vec{B}} \hat{F}.$$

(I emphasize that \hat{F} is not the matrix corresponding to the Hodge-dual of F_A in the given metric with signature $(+ + + -)$.) On the other hand, representing both the 1-forms and

vectors as columns so that in particular

$$\nabla_{3+1} f = \begin{bmatrix} f_x \\ f_y \\ f_z \\ -f_t \end{bmatrix} \quad \text{and} \quad \delta F_A = \begin{bmatrix} E_{1,t} + B_{2,z} - B_{3,y} \\ E_{2,t} + B_{3,x} - B_{1,z} \\ E_{3,t} + B_{1,y} - B_{2,x} \\ E_{1,x} + E_{2,y} + E_{3,z} \end{bmatrix},$$

one checks directly that

$$F_A(\nabla_{3+1} f, \cdot) = F \nabla_{3+1} f.$$

This enables us to rewrite Eq. (5) in the form

$$F^{-1} \delta F_A = \nabla_{3+1} \ln(f). \tag{6}$$

It is perhaps worthwhile to realize that in this context F is a fiberwise-linear mapping from the tangent bundle to the cotangent bundle. Here one assumes $f > 0$ a.e. This conforms with the principle that one will be consistently looking for *strong* solutions so that in particular f may always be replaced with $|f|$ in (1)–(3). Next one recalls that on one hand the first part of the NM (1) is identical with the analogous part of the classical Maxwell equations and it encodes the Faraday’s law of magnetic induction and the fact that there are no spatially extensive magnetic charges. This gives the first four (scalar) equations below, namely (7) and (8). On the other hand, by direct multiplication and regrouping in (6), one obtains four further scalar equations that happen to radically modify the Ámpere law. Written in the familiar three-space vector notation, the NM assume the form

$$\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0, \tag{7}$$

$$\nabla \cdot \vec{B} = 0, \tag{8}$$

$$\left(\frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{B} \right) \times \vec{E} + (\nabla \cdot \vec{E}) \vec{B} = -(\vec{E} \cdot \vec{B}) \nabla \ln f, \tag{9}$$

$$\left(\frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{B} \right) \cdot \vec{B} = -(\vec{E} \cdot \vec{B}) \frac{\partial}{\partial t} \ln f, \tag{10}$$

$$\left(\frac{\partial^2}{\partial t^2} - \Delta \right) f + a(|\vec{B}|^2 - |\vec{E}|^2) f = \nu f, \tag{11}$$

provided $\vec{E} \cdot \vec{B} \neq 0$. In fact, also the case when $\vec{E} \cdot \vec{B} = 0$ is worth our attention and will be discussed below. As much as one should avoid indulging in formal manipulations of formulas, here the advantage of having the equations rewritten in several equivalent forms is that they all lead to the discovery of new types of solutions, the existence of which would be otherwise obscured by notation. This will become more evident in the following sections.

As already mentioned in Section 1, I postulate the following physical interpretation: f is the spatially varying filling factor—a notion central to the modern composite particle

theories. In fact, the canonical microscopic theory interpretation of the filling factor is valid in two spatial dimensions only, in which case it signifies the ratio of the number of quanta of the ambient magnetic field to the number of electrons in the first Landau level, cf. [7,17]. Moreover, the microscopic theory offers no hints as to the existence and relevance of an analogous notion in the three-space. A description of the interaction of the electromagnetic field with fermions in the first Landau level provided by the equations above is valid in the mesoscopic scale. Here, as one ‘zooms out’ from the microscopic scale, the filling factor is neither a rational number nor is it a constant anymore. In fact, as it has been communicated in previous papers the spatially varying filling factor may assume the form of a vortex lattice, cf. [13]. For the time being, this point of view is validated by the well known analogy between the Quantum Hall Effect and the High Temperature Superconductivity and it awaits experimental confirmation. Moreover, the NM extend the notion of the filling factor to three spatial dimensions. However, as we will see below, the presence of the filling factor introduces an especially interesting modification of the laws of Electromagnetism only if the three-space comes equipped with a codimension one foliation. This latter fact makes it possible to talk about Landau levels in a certain sense, anyhow. Finally, f can be completely eliminated from the NM in $3 + 1$ -dimensions. (In general, this requires that the first cohomology group of the spacetime vanishes.) In that case the NM can be written in the f -free form

$$dF_A = 0, \quad d(F^{-1}\delta F_A)^\# = 0, \tag{12}$$

$$\delta(F^{-1}\delta F_A)^\# - |F^{-1}\delta F_A|^2 + a|F_A|^2 - \nu = 0, \tag{13}$$

where $\#$ is the isomorphism of the tangent and the cotangent bundles given by the metric. Indeed, under the assumption of vanishing first de-Rham cohomology, Eq. (6) is equivalent to its integrability condition (12). Moreover, since the last scalar equation of the system can be written in terms of $d \ln f$ in the form

$$\delta d \ln f - |d \ln f|^2 + a|F_A|^2 - \nu = 0.$$

Eq. (6) also implies (13). Computation of the symbol shows that the system obtained in this way is nonhyperbolic—in fact its degeneracy is of higher order. Thus, this form of the NM appears impractical for any mathematical work, and an introduction of the dimensionless scalar f is necessary also from the point of view of analysis. Nevertheless, as indicated in Section 1 and the discussion above, physical implications of the existence of an f -free form of the NM are important.

3. Geometry behind the equations

The geometrical arena of the Maxwell equations consists of a spacetime, say N , and a principal $U(1)$ -bundle, say P , stack up above N . In addition, it seems any description of the interaction of the electromagnetic field with fermions requires, at least within this framework, a principal connection, i.e. a smooth (at least a.e.) distribution of horizontal planes that is invariant with respect to the circle action. This distribution can be written as $\ker A = \ker fA$ for $f \neq 0$. In addition, if $U(1)$ is to remain the elemental symmetry group of

Electromagnetism, then f must be constant along the fibers so that it effectively descends to a function on N . In particular, within this dictionary one can construct a Kaluza–Klein metric on P , which is given by

$$\mu_A(X, Y) = g(\pi_*X, \pi_*Y) + aA(X)A(Y),$$

where the unit of $a > 0$ must be $[T^{-2} m^{-2}]$ if the unit of length on P is to be $[m]$ and the unit of $F_A = dA$ is to remain, say, $[T]$. Let us say the corresponding Laplace–Beltrami operator on forms is then $\Delta_{\mu_A} = \Delta_A$. Calculation shows that the condition

$$\Delta_A(fA) = \nu fA,$$

is equivalent to the system of Eqs. (1)–(3), cf. [11].

4. Exotic duality

For the sake of discussion in this section, consider the NM on either a Lorentzian or a Riemannian four-manifold as the metric signature plays a secondary role. In particular, it is preferable to replace the \square -notation with the Δ -notation. Assume for the sake of simplicity that the second cohomology group of the manifold is trivial. Omitting the constant a , write the system one more time in the form

$$\delta(f dA) = 0, \tag{14}$$

$$-\Delta f + |dA|^2 f = \nu f. \tag{15}$$

Since (14) implies $d(f \star dA) = 0$, one has $f \star dA = d\tilde{A}$ so that

$$dA = \pm \frac{1}{f} \star d\tilde{A}, \tag{16}$$

and the new form \tilde{A} satisfies a *dual* system of equations

$$\delta\left(\frac{1}{f} d\tilde{A}\right) = 0, \tag{17}$$

$$-\Delta f + |d\tilde{A}|^2 \frac{1}{f} = \nu f. \tag{18}$$

This is a functional transform reminiscent of the Fourier or the Backlund transforms, notwithstanding the fact that all transforms are somewhat reminiscent of one another. In particular, the resulting dualistic perspective has the expected property that trivial solutions of one of the systems lead to more complex solutions of the dual system. To illustrate the idea, let me now present a few examples of dual solutions on R^4 with either the Euclidean or the Minkowski metric as specified in the discussion.

Example 1. Let the metric be Euclidean and take $dA = E dz \wedge dt$, $E = \text{const}$, and $f = f(x, y)$. Eq. (14) is automatically satisfied and (15) assumes the form $-f_{xx} - f_{yy} =$

$(v - E^2)f$ so that $f = \cos(k_1x + k_2y + \alpha)$ for $k_1^2 + k_2^2 = v - E^2$ solves the problem. Now $d\tilde{A} = \pm f(x, y) dx \wedge dy$ and it satisfies Eqs. (17) and (18).

Example 2. Departing for a while from the assumption of vanishing second cohomology, let us reinterpret the previous example on a four-torus assuming periodicity of coordinates (x, y, z, t) with period 2π . Note that the first bundle is necessarily nontrivial as the cohomology class $[dA] \neq 0$. Let us allow the function f drop its dependence on y so that, say, $f = \cos x$, provided the ‘right choice’ of v has been made. Now, $d\tilde{A} = d(\sin x dy)$ is an exact form so the second bundle is topologically trivial.

Example 3. Consider $dA = B dx \wedge dy$ and $f = f(z, t)$ so that (14) is satisfied. Let us now look at the metric with signature $(+++-)$ so that (15) means $f_{tt} - f_{zz} = (v - B^2)f$. The general solution of this equation is a standing wave with variable amplitude. This pattern is inherited by $d\tilde{A} = f(z, t) dz \wedge dt$ (up to the sign again) which satisfies (17) and (18).

Example 4. Let us for a change begin on the other side and take, say, $d\tilde{A} = e dz \wedge dt$ and $f = f(x, y)$. Again, the first Eq. (14) is automatically satisfied while (15) becomes $-f_{xx} - f_{yy} + e^2/f = \nu f$. As explained in [13] (see also remarks at the end of Section 6 below) apart from the trivial constant solution, this problem also has a solution in the form of a vortex lattice. In the latter case $dA = f(x, y) dx \wedge dy$ satisfies (14) and (15) and represents static magnetic flux tubes.

I emphasize that only the vector potential A and the filling fraction variable f that appear in the first set of equations have physical interpretation. Reassuringly, the presence of a nontrivial f in Examples 1 and 3 did not contribute anything unexpectedly strange to the constant electric and magnetic fields in these examples, while it ‘introduced’ flux tubes in Example 4. Although one could consider similar interpretation of the transformed vector potential \tilde{A} , just as one can for any $U(1)$ -connection, I feel this is uncalled for and would probably be unjustifiable at this point. Nevertheless, the existence of the transform is a remarkable fact whose possible applications to four-manifolds will be explored more thoroughly in the future. In a way, this new duality is a generalization of the regular Hodge-star duality that may be compared to the projective generalization of the Euclidean reflection. This analogy may be justified in the following way. Projective duality is induced by a fixed quadratic form. What is the NM analog of that object? Introduce notation $\varphi = \ln f$. A direct calculation shows that (14) and (15) may be written in the form of a system of quadratic equations

$$\delta dA + \star(d\varphi \wedge \star dA) = 0, \tag{19}$$

$$-\Delta\varphi - |d\varphi|^2 + |dA|^2 - \nu = 0. \tag{20}$$

This form of the equation has one other advantage. Suppose one has found a solution (A, φ) of (19) and (20). One can now use gauge invariance of the equations in the following way. Let χ be a solution of the equation

$$\delta d\chi = -\delta A.$$

The existence of a solution χ follows from the Fredholm alternative when the metric is positive definite, and it amounts to solving a linear wave equation in a Lorentzian metric. One can now replace A with $A + d\chi$ (and denote the resulting form by A again). In the new gauge $\delta A = 0$, so that A in fact satisfies

$$\Delta A + \star(d\varphi \wedge \star dA) = 0. \quad (21)$$

The system that consists of (20) and (21) is either quasilinear elliptic or hyperbolic, depending on the metric. Solving the latter system may not be helpful at all in finding solutions of the original (19) and (20), since one cannot guarantee that a solution satisfies the Lorentz gauge condition $\delta A = 0$. However, solutions of (19) and (20) a fortiori satisfy (20) and (21) so that in particular they will obey all a priori estimates on the solutions of, say, quasilinear hyperbolic systems. In particular, this point of view may justify the claim that the phenomena described in this paper shed some light on the complex nature of quasilinear systems of PDE of certain types in general.

5. Charge transport and charge stripes

I will now take full advantage of the (7)–(11) form of the NM. In analogy to the electromagnetic wave in vacuum, that one recalls is counted among the solutions of this system, one wants to look for a solution with $\vec{E} \cdot \vec{B} = 0$. In the end I will check that the new solution of (7)–(11) in fact satisfies (1)–(3) which is not a priori guaranteed. Make an Ansatz

$$\vec{B} = B_1 \frac{\partial}{\partial x} + B_2 \frac{\partial}{\partial y}, \quad \vec{E} = e \left(-B_2 \frac{\partial}{\partial x} + B_1 \frac{\partial}{\partial y} \right), \quad (22)$$

where e , B_1 and B_2 are a priori functions of (x, y, z, t) that are smooth a.e. and neither one of them vanishes identically. As an immediate consequence, one obtains that (7) and (8) are equivalent to

$$B_{1,t} = (eB_1)_{,z}, \quad (23)$$

$$B_{2,t} = (eB_2)_{,z}, \quad (24)$$

$$(eB_1)_{,x} + (eB_2)_{,y} = 0, \quad (25)$$

$$B_{1,x} + B_{2,y} = 0, \quad (26)$$

which implies

$$e_{,x}B_1 + e_{,y}B_2 = 0. \quad (27)$$

On the other hand, (9) and (10) are equivalent to

$$(B_{2,x} - B_{1,y})eB_1 = (eB_2)_{,x}B_1 - (eB_1)_{,y}B_1, \quad (28)$$

$$(B_{2,x} - B_{1,y})eB_2 = (eB_2)_{,x}B_2 - (eB_1)_{,y}B_2, \quad (29)$$

$$-(eB_2)_{,t} + B_{2,z})B_1 + ((eB_1)_{,t} - B_{1,z})B_2 = 0. \quad (30)$$

Eqs. (23), (24) and (30) implies that e is in fact constant

$$e = \pm 1. \tag{31}$$

Using (23) and (24) again, one obtains

$$B_1 = B_1(x, y, t + ez), \quad B_2 = B_2(x, y, t + ez).$$

In particular, \vec{B} and \vec{E} are not compactly supported. At this point, the only condition left a priori unfulfilled is the vanishing divergence condition. Thus, all Eqs. (23)–(30) above are satisfied if there is a function $\psi = \psi(x, y, t + ez)$ such that

$$B_1 = -\psi_y(x, y, t + ez), \quad B_2 = \psi_x(x, y, t + ez). \tag{32}$$

Defining the electric and magnetic fields by (22) with $e = \pm 1$, so that in particular $|\vec{E}| = |\vec{B}|$, and choosing f that satisfies the linear wave Eq. (11), one obtains a solution of (7)–(11).

However, physical solutions must in addition satisfy the a priori more restrictive system (1)–(3). Consider F_A as given in (4). Eq. (1) is satisfied automatically since it is equivalent to (7) and (8). On the other hand, (2) becomes

$$(fB_2)_{,x} - (fB_1)_{,y} = 0, \tag{33}$$

$$(feB_2)_{,t} - (fB_2)_{,z} = 0, \tag{34}$$

$$-(feB_1)_{,t} + (fB_1)_{,z} = 0, \tag{35}$$

$$-(fB_1)_{,y} + (fB_2)_{,x} = 0. \tag{36}$$

Now, (34) and (35) imply via (32) that

$$f = f(x, y, t + ez).$$

In particular, $f_{,tt} - f_{,zz} = 0$. Thus, (1)–(3) has been reduced to the following system of two equations:

$$-f_{,xx} - f_{,yy} = \nu f, \tag{37}$$

$$(f\psi_{,x})_{,x} + (f\psi_{,y})_{,y} = 0. \tag{38}$$

The first equation above admits three types of classical solutions. Namely,

$$f = \begin{cases} A(t + ez) \ln(x^2 + y^2), & \nu = 0, \\ A(t + ez) \cos(k_1x + k_2y + \alpha(t + ez)), & \nu = k_1^2 + k_2^2, \\ A(t + ez) \exp(k_1x + k_2y), & \nu = -k_1^2 - k_2^2. \end{cases} \tag{39}$$

Observe that each solution effectively depends on one harmonic variable in the (x, y) -domain—either, say, $u = k_1x + k_2y$ or $u = \ln r^2 = \ln(x^2 + y^2)$. Thus, Eq. (38) is satisfied if

$$\psi_u = \frac{C(t + ez)}{f(u, t + ez)},$$

for an arbitrary function C of one variable. Therefore, in view of (32) one obtains three types of solutions (redefining C)

$$[B_1, B_2] = \begin{cases} \frac{C(t + ez)}{r^2 \ln r^2} [-y, x] \\ C(t + ez) \sec(k_1x + k_2y + \alpha) [-k_2, k_1] \\ C(t + ez) \exp(-k_1x - k_2y) [-k_2, k_1] \end{cases}, \tag{40}$$

in correspondence with (39). Since one is looking for strong solutions, one has the freedom to cut off pieces of the classical solutions (by restricting the domain) and to put them back together. In this way, one obtains solutions that are either continuous or have jump discontinuities but may be guaranteed to remain bounded. Last but not least, it is physically correct to interpret the divergence of the electric field as charge ρ and $-(\partial/\partial t)\vec{E} + \nabla \times \vec{B}$ as the electric current. One checks that for solutions as above the (x, y) -component of current vanishes while the z -component j is equal to $-e\rho$. More precisely, one obtains that piecewise

$$e\rho = -j = \begin{cases} \frac{4C(t + ez)}{r^2 \ln^2 r^2} \\ \nu C(t + ez) \sec(k_1x + k_2y + \alpha) \tan(k_1x + k_2y + \alpha) \\ -\nu C(t + ez) \exp(-k_1x - k_2y) \end{cases}, \tag{41}$$

in correspondence with (39) and (40). In addition to the piecewise smooth distribution of charge, one should include charge concentrated on singular surfaces where the electric field has jump discontinuities as indicated by the distributional derivative $\nabla \cdot \vec{E}$. Therefore, charge is transported along the z -axis with the speed $e = \pm 1$ and without resistance as the vector of current is perpendicular to the electric field. Charge is mostly concentrated along *charge stripes* where the electric and magnetic fields have singularities. The net current depends on the particular choice of a (strong) solution. Of course, the theory does not tell us how to solve the practical problem of electronics—namely, how to create conditions for a particular function $C = C(t + ez)$, constant ν and a desired mosaic of singularities to actually occur in a physical system.

6. Static solutions and magnetic flux tubes

The classical Maxwell equations admit static solutions of two types only: the uniform field solutions, and the unit charge or monopole-type solutions, as well as superpositions of these fundamental types of solutions. As we will see below, the nonlinear theory encompasses a larger realm including the magnetic flux tube type and the charge stripe type solutions. These additional configurations require nonlinearity and cannot be superposed, which gives them more rigidity. In the next section we will see what can be said about the variety of such solutions, while in this section I will only display a single example of this type. Apart from the applicable goal, the idea is to present an example that possesses all the essential features of the general class of solutions yet the required calculation is free of more subtle geometric technicalities.

Time-independent solutions of the NM possess physical interpretation only if they satisfy the equations in the classical sense almost everywhere. Assuming that all fields are independent of time (7)–(11) takes on the form

$$\nabla \times \vec{E} = 0, \tag{42}$$

$$\nabla \cdot \vec{B} = 0, \tag{43}$$

$$-(\nabla \times \vec{B}) \times \vec{E} + (\nabla \cdot \vec{E})\vec{B} = -(\vec{E} \cdot \vec{B})\nabla \ln f, \tag{44}$$

$$(\nabla \times \vec{B}) \cdot \vec{B} = 0, \tag{45}$$

$$-\Delta f + a(|\vec{B}|^2 - |\vec{E}|^2)f = \nu f, \tag{46}$$

under the assumption that $\vec{E} \cdot \vec{B} \neq 0$ a.e. Adopt an Ansatz that the integral surfaces of the planes perpendicular to the field \vec{B} are flat, say,

$$\vec{B} = b(x, y) \frac{\partial}{\partial z}.$$

One easily checks that Eqs. (43) and (45) are satisfied. Assume in addition that the electric field is potential, i.e.

$$\vec{E} = \nabla \psi(x, y, z), \text{ where } \psi_z \neq 0 \text{ a.e.,}$$

so that (42) is satisfied. Remembering notation $\varphi = \ln f$, one calculates directly that

$$(\nabla \times \vec{B}) \times \vec{E} = -\psi_z b_x \frac{\partial}{\partial x} - \psi_z b_y \frac{\partial}{\partial y} + (\psi_x b_x + \psi_y b_y) \frac{\partial}{\partial z},$$

while

$$(\nabla \cdot \vec{E})\vec{B} = \Delta \psi b \frac{\partial}{\partial z},$$

and

$$(\vec{E} \cdot \vec{B})\nabla \varphi = b \psi_z \left(\varphi_x \frac{\partial}{\partial x} + \varphi_y \frac{\partial}{\partial y} + \varphi_z \frac{\partial}{\partial z} \right).$$

Thus, Eq. (44) is equivalent to the following system of three equations:

$$\psi_z (b \varphi_x + b_x) = 0,$$

$$\psi_z (b \varphi_y + b_y) = 0,$$

$$b \Delta \psi - \psi_x b_x - \psi_y b_y - b \psi_z \varphi_z = 0,$$

and since $\psi_z \neq 0$ one obtains from the first two equations

$$\varphi(x, y, z) = \varphi_1(x, y) + \varphi_2(z) \quad \text{and} \quad b = \beta \exp(-\varphi_1),$$

while the third equation assumes the form

$$\Delta \psi + \nabla \psi \cdot \nabla \varphi = 0. \tag{47}$$

At this point the NM have been reduced to the system of just two scalar Eq. (46) and (47). Denote $f_1 = \exp \varphi_1$ and $f_2 = \exp \varphi_2$ and assume in addition

$$\psi = \psi(z),$$

so that

$$\psi'(z) = \epsilon \exp(-\varphi_2) = \frac{\epsilon}{f_2}.$$

It now follows from (46) and (47) that the triplet

$$\vec{B} = \frac{\beta}{f_1(x, y)} \frac{\partial}{\partial z}, \quad \vec{E} = \frac{\epsilon}{f_2(z)} \frac{\partial}{\partial z}, \tag{48}$$

and

$$f(x, y, z) = f_1(x, y) f_2(z), \tag{49}$$

is a solution of the NM if only f_1 and f_2 satisfy a decoupled system of semi-linear elliptic equations

$$-f_2''(z) - \frac{\epsilon^2}{f_2(z)} = v_2 f_2(z), \tag{50}$$

$$-\Delta f_1(x, y) + \frac{\beta^2}{f_1(x, y)} = v_1 f_1(x, y). \tag{51}$$

At this point, I would like to emphasize one more time that in a field theory one looks for *strong* solutions, i.e. solutions that satisfy equations in the classical sense almost everywhere. Typically, such solutions are smooth except for singularities supported on a union of closed submanifolds. Furthermore, geometrically invariant derivatives of the resulting fields in the distributional sense signify charges. With this understood, let us briefly turn attention to Eq. (50). One wants to avoid holding the reader hostage to the formal analysis of this elementary equation which might be somewhat distracting. Thus, I have chosen to briefly describe the solutions qualitatively leaving aside technical details that can be easily reconstructed aside by the reader. First, one notes that if $v_2 > 0$ then a solution is concave, while for $v_2 < 0$ it will be convex for large values where $f_2^2 > -\epsilon^2/v_2$. Assuming formally that f_2 is a function of f_2' (piecewise), one reduces (50) to the first-order equation

$$\frac{df_2}{dz} = \pm \sqrt{c - v_2 f_2^2 - \epsilon^2 \ln f_2^2}.$$

Thus, there are essentially two types of positive solutions, depending on the actual values of constants c, ϵ, v_2 . The first type includes solutions that assume value 0 at a certain point z_0 and increase monotonously to infinity as $z \rightarrow \infty$ as well as the symmetric solutions defined between $-\infty$ and some point, say z_0 again, where they reach 0. These solutions require $v_2 < 0$ and they asymptotically look like $\exp(\pm(-v_2)^{1/2}z)$. One can use both branches in order to put together a strong solution that forms a cusp or a jump discontinuity at z_0 . The second type consists of solutions that are concave, rise to the highest peak at $f_2 = m$, when $c - v_2 f_2^2 - \epsilon^2 \ln f_2^2 = 0$, and fall off to 0 on both sides in finite time while being

differentiable in-between. Selecting the constants and combining both types of solutions piecewise segment-by-segment one obtains strong solutions f_2 that in turn provide electric fields according to formula (48).

Since, with the exception of the trivial constant solution, there are no global smooth solutions, one concludes that either \vec{E} is constant or there exist charge stripes located at planes $z = \text{const}$ where $f_2(z)$ has singularities. The distributional derivative is in each case equal to the Dirac measure concentrated at $z = \text{const}$ as above and scaled by the size of the jump, and classical derivatives on both sides of the singularity. Even in absence of a jump, the charge will switch from negative to positive thus forming what can be amenably called a charge stripe. An example of this is shown in Fig. 1.

It is much more difficult to figure out solutions of the second equation. I refer the reader to [13] for a more thorough analysis, while here I will just briefly summarize my previous findings. Solutions of Eq. (51) correspond to critical points of the functional

$$L(f_1) = \frac{(1/2) \int |\nabla f_1|^2 + \beta^2 \int \ln(f_1)}{\int f_1^2},$$

which is neither bounded below nor above, so that one is looking at the problem of existence of local extrema. The equation always admits a trivial constant solution. But, as it is shown

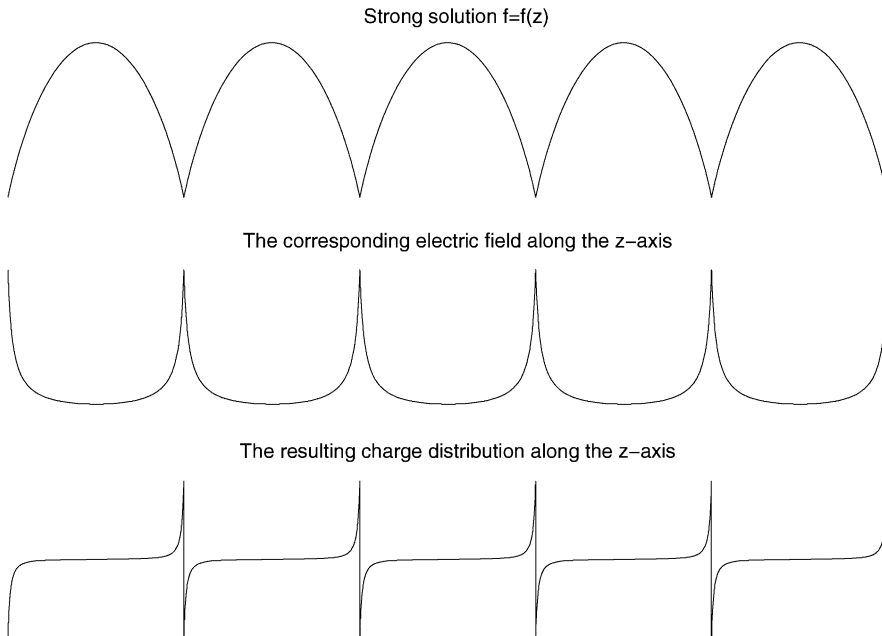


Fig. 1. An example of a strong solution of (50). $f = f_2(z)$ is a positive function, the electric field is given by formula (48). The resulting charge distribution is obtained by evaluating $\nabla \cdot \vec{E}$. (In general, $\nabla \cdot \vec{E}$ is understood in the distributional sense.) Charge is concentrated along certain $z = \text{const}$ planes. This is the basic appearance of charge stripes—intertwining concentrations of positive and negative charges. (One should compare this static picture with the description of moving charge stripes in Section 5.)

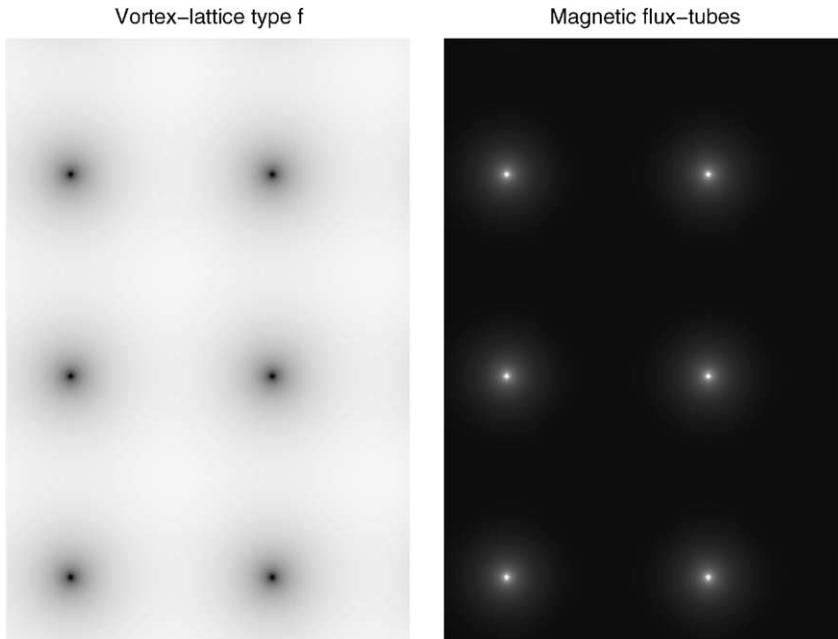


Fig. 2. The luminance graph of $f = f_1(x, y)$ that solves (51). The corresponding magnetic field on the right is obtained via (48).

in [13], it also possesses nontrivial vortex lattice solutions. More precisely, if β is larger than a certain critical value then there is a nonconstant doubly periodic function f which satisfies the finite difference version of (51) everywhere except at a periodic lattice of isolated points, one point per each cell. In this way, a lattice of flux tubes, cf. Fig. 2, emerges as a solution of the NM. For the time being, the proof of this fact relies on finite-dimensionality essentially, and does not admit a direct generalization to the continuous-domain case. However, physical parameters, like $\int f_1^2$ and β , are asymptotically independent of the density of discretization. Thus, I conjecture existence of the continuous-domain solutions that satisfy the equation a.e. in the classical sense and retain the particular vortex morphology. Presently, the essential obstacle to proving this conjecture is lack of a regularity theory for the discrete vortex solutions. The proof in [13] is carried out in the (discretized) torus setting. One believes that vortex type solutions exist on any closed (orientable) surface.

7. Topological quantum numbers

Every gauge theory comes equipped with an associated set of topological invariants—usually characteristic classes of the bundles used to introduce the gauge field. Articles [4–6] teach us how such topological invariants may be manifested in an electronic system as observable quantum numbers. The Nonlinear Maxwell Theory is naturally equipped

with two kinds of topological invariants. On one hand, one has the first Chern class of the original $U(1)$ -bundle. Additionally, we will see below that in the case of static solutions the NM give us an additional set of invariants defined directly by the foliated structure of the underlying three-manifold. (In the discussion below, I generally assume for the sake of simplicity that M is a closed orientable manifold unless stated otherwise.) In this section I will make an effort only to identify rather than exploit to the fullest the geometric and topological ramifications of this nonlinear theory of Electromagnetism. To gain some initial impetus, let us be guided by the following question: ‘What are the necessary and sufficient conditions on a Riemannian three-manifold M for the NM to admit a separation of variables of the type seen in the previous section, i.e. for the Eq. (51) to decouple so that its solutions will generate magnetic flux tube type solutions on M ’?

A question of this type is typical in algebraic topology where one is asking about global obstructions to the presence of certain algebraic factorization properties of analytic objects, like linear differential equations as it is the case for, say, the de-Rham cohomology groups. In our case, the equations are nonlinear, but the principle remains the same. The importance of these questions for practical issues of Electromagnetism is twofold. First, one wants to know how big is the set of possible configurations—especially in the absence of the superposition principle. Secondly, I believe the topological invariants displayed below are directly on target in an effort to explain and describe the nature of certain rigid structures, like the Quantum Hall Effects, that physically occur in electronic systems.

First, it needs to be emphasized that the static field equations I want to consider, i.e. the equations that descend from the four-dimensional spacetime via time-freezing coefficients, are distinct from the Eqs. (1)–(3) considered directly on a three-manifold. Secondly, the Eqs. (42)–(46) are only valid on a Euclidean space. The geometry behind these equations is easier to identify when they are rewritten in an invariant form that can be considered on any three-manifold in a coordinate independent setting.

Fix a Riemannian metric on M with scalar product $\langle \cdot, \cdot \rangle$ extended to include measuring differential forms. Denote by B and E the forms dual to the magnetic and electric field vectors; recall notation $\varphi = \ln f$ and put $\alpha = 1$. The static NM assume the form

$$dE = 0, \tag{52}$$

$$\delta B = 0, \tag{53}$$

$$\star(\star dB \wedge E) + (\delta E)B = -\langle E, B \rangle d\varphi, \tag{54}$$

$$dB \wedge B = 0, \tag{55}$$

$$\Delta\varphi + |d\varphi|^2 + |E|^2 - |B|^2 + \nu = 0. \tag{56}$$

Eq. (55) is the familiar Frobenius condition on integrability of the distribution of planes given by $\ker B$. One always assumes B is nonsingular a.e. so that the distribution is a priori also defined a.e. For convenience, it is assumed throughout this section that the foliation determined by $\ker B$ is smooth. (It is quite clear that for the flux tube type solutions the distribution extends through the singular points and is defined everywhere. At this stage, however, it is hard to make a formal argument to this effect, hence the a priori assumption.) The condition of smoothness implies that the three-manifold M must have vanishing Euler

characteristic. In particular, singular foliations, some of which may be associated with other types of solutions of the NM, are excluded from the discussion below.

It follows that there is a 1-form α , known as the Godbillon–Vey form, such that

$$dB = \alpha \wedge B.$$

This form is not defined uniquely. However, as is well known, $d(\alpha \wedge d\alpha) = 0$ and the Godbillon–Vey (GV) cohomology class

$$[\alpha \wedge d\alpha]_{H^3(M)},$$

is uniquely defined. On a three-manifold this class can be evaluated by integration resulting in a GV-number

$$Q = \int_M \alpha \wedge d\alpha.$$

This invariant poses many interesting questions that have not been fully resolved by geometers yet. Below, I will justify two observations. First, the condition of existence of the magnetic flux tube solutions imposes both local and global restrictions on the foliation. Second, magnetic flux tube solutions exist in topologically nontrivial situations with nonzero GV-number Q . This is formally summarized in the two propositions that follow. They are far from the most general statements that can be anticipated in this direction, but are also nontrivial enough to suggest a conjecture regarding *quantization* of the GV-number that I will formulate following [Proposition 1](#).

Consider a priori a foliation given by $\ker B$ locally. First, one introduces a local coordinate-patch (x, y, z) such that the foliation is given by the (x, y) -planes on which $z = \text{const}$, and let ω be the unit-length normal 1-form, i.e. $\omega = dz/|dz|$. In particular

$$B = \beta(x, y, z)\omega,$$

for a certain function $\beta = \beta(x, y, z)$. Let $\gamma = g(x, y, z) dx \wedge dy \wedge \omega$ denote the volume element on a leaf, so that one has $d\star B = d(\beta g(x, y, z) dx \wedge dy)$. Now, [Eq. \(53\)](#) implies that there is a function $\chi = \chi(x, y)$ such that

$$\beta(x, y, z) = \frac{\chi(x, y)}{g(x, y, z)}.$$

A somewhat tedious calculation analogous to that in the previous section shows that when B is of this form, the whole system [\(53\)–\(56\)](#) is reduced to

$$\left(\ln \frac{\chi(x, y)}{g(x, y, z)} \right)_x = -\varphi_x, \quad \left(\ln \frac{\chi(x, y)}{g(x, y, z)} \right)_y = -\varphi_y, \tag{57}$$

$$\delta E + \langle E, d\varphi \rangle = 0, \tag{58}$$

$$\Delta\varphi + |d\varphi|^2 + |E|^2 - \left(\frac{\chi(x, y)}{g(x, y, z)} \right)^2 + \nu = 0. \tag{59}$$

Observe that in view of [\(57\)](#) in order to obtain factorization

$$\varphi(x, y, z) = \varphi_1(x, y) + \varphi_2(z), \tag{60}$$

it is necessary that

$$g = g(x, y), \quad (61)$$

i.e. a priori dependence of g on z is dropped. Now, if (60) and (61) hold, then Eqs. (58) and (59) can be (locally) decoupled—indeed, it suffices to select E of the form $E = e(z) dz$. In this case, one also has that $\chi/g = b \exp(-\varphi_1)$ for a constant b and therefore, φ_1 satisfies the vortex lattice equation:

$$\Delta_{x,y}\varphi_1 + |d\varphi_1|^2 - b^2 \exp(-2\varphi_1) + \nu = 0. \quad (62)$$

However, (61) implies that the mean curvature h of a leaf vanishes. Indeed, by definition

$$h = \delta \left(\frac{1}{|B|} B \right) = -\star d(g(x, y) dx \wedge dy) = 0.$$

These local coordinate-patch considerations lead us to the following.

Proposition 1. *For the existence of flux tube type solutions—in the sense of existence of factorization (60) and decoupling of Eq. (62)—it is necessary that the foliation given by $\ker B$ be taut, i.e. the mean curvature of leaves must vanish. In particular $\pi_1(M)$ must be infinite.*

Proof. The first part has been shown above. The second part follows from a result of Sullivan [14] that he deduced from the result of Novikov on the existence of a closed leaf that is a torus (cf. [8,16] for additional general material and references). \square

In particular, there are no flux tube type solutions of the NM that would conform with the Reeb foliation [9]. This is a practical issue since the Reeb foliation exists on a solid torus, so that in principle it might be observed experimentally which would be inconsistent with the theory at hand. This fact is also interesting for another reason. Namely, according to the celebrated theorem by Thurston [15] each real number may be realized as the GV-number for a certain codimension one foliation on the three-sphere S^3 . The known proof of this result uses the Reeb foliation in an essential way. I do not know if this fact is canonical, i.e. if the presence of the Reeb foliation is necessary for the result to hold, but if it turns out to be so then excluding the Reeb foliation from the game should result in a reduction of the range of the GV-number, possibly to a discrete subset of the real line. In such a case, the resulting set of the GV-numbers accompanying flux tube type solutions of the NM would also be discrete. This is consistent with my expectation that these invariants must be related with (both the integer and the fractional) Quantum Hall Effects. Future research should bring a resolution of this problem.

Another observation is that the factorization given by (60) and (62) does exist in topologically nontrivial situations. More precisely, I want to consider solutions of the NM on $\mathrm{PSL}(2, R)$ and its compact factors. These three-manifolds are equipped with interesting codimension one foliations known as the Roussarie foliation [10]. Let the Lie algebra $\mathrm{PSL}(2, R)$ be given by

$$[X, Y^+] = Y^+, \quad [X, Y^-] = -Y^-, \quad [Y^+, Y^-] = 2X.$$

Pick a metric on $\text{PSL}(2, R)$ in which the corresponding left-invariant vector fields $X, Y^+,$ and Y^- are orthonormal and let $\mu, v^+,$ and v^- be the corresponding dual 1-forms. One checks directly that

$$dv^- = \mu \wedge v^-.$$

So that the distribution $\ker v^-$ is integrable and μ is the GV-form of the resulting foliation. In particular, one can introduce local coordinates (x, y, z) such that the vector ∂_z is parallel to Y^- , while X and Y^+ are tangent to the (x, y) -planes (on which $z = \text{const}$).

This foliation descends to compact factors of $\text{PSL}(2, R)$ that can each be identified with T_1M_g —the total space of the unit tangent bundle of the hyperbolic Riemannian surface of genus g that depends on our choice of the co-compact subgroup acting on $\text{PSL}(2, R)$ by isometries. Moreover, the GV-integrand $\mu \wedge d\mu$ is proportional to the natural volume form on the three-manifold. As a result of this, the corresponding GV-numbers

$$Q = \int_{T_1M_g} \mu \wedge d\mu = -2 \text{Vol}(M_g),$$

assume values in a discrete set. I want to look for solutions of the NM that satisfy the Ansatz

$$B = \beta v^-. \tag{63}$$

In particular (the Frobenius) Eq. (55) is satisfied automatically. Moreover, since $d\star B = (Y^- \beta)\mu \wedge v^+ \wedge v^-,$ Eq. (53) implies

$$\beta = \beta(x, y). \tag{64}$$

As before, one checks that (54) implies (58) as well as $X\varphi = -X \ln \beta$ and $Y^+\varphi = -Y^+ \ln \beta.$ In consequence, one again has (60) and assuming $E = e(z) dz$ as before one obtains (62). In consequence, the following holds true.

Proposition 2. *The Roussarie foliations on $\text{PSL}(2, R)$ and its compact factors satisfy the factorization condition for the existence of magnetic flux tube type solutions in the sense that the tangent distribution can be expressed as $\ker B$ a.e. and one can reduce the NM to the form (60)–(62).*

The point is, of course, that due to homogeneity of the foliated structure a solution φ_1 of (62) obtained on one leaf of the foliation gives rise to a globally defined solution that satisfies the NM over the whole manifold with, say, $\varphi_2 = 0$ and $E = 0.$ This ‘algebraic’ result will become a rigorous PDE global existence type result as soon as the theory of vortex lattice solutions is worked out analytically, and not just in the form of a (asymptotically stable) discrete theory. Alternatively, the quantized version of the NM may also be used in order to provide the existential part of this geometric approach.

In a similar way one can obtain factorization (60) and (62) for other foliations, like the natural foliation on say $S^2 \times S^1.$

8. More on the physical framework of the NM

It is natural to ask if the NM descend from a Lagrangian functional depending on the two variables A and f , say $\Phi(A, f)$, via the Euler–Lagrange calculus of variations. The answer is negative as one can easily see considering that in general a gradient must pass the second derivative test:

$$\frac{\delta^2}{\delta A \delta f} \Phi = \frac{\delta^2}{\delta f \delta A} \Phi.$$

This condition cannot be satisfied by the expressions in (1)–(3) viewed as the gradient, say $((\delta/\delta A)\Phi, (\delta/\delta f)\Phi)$, of an unknown functional Φ . This suggests that the NM may constitute just a part of a broader theory that would encompass additional physical fields. In other words, the Eqs. (1)–(3) would have to be coupled to some other equations via additional fields. In addition, such coupling would have to induce only a very small perturbation of the present picture that one believes is essentially accurate. Such possibilities may become more accessible in the future.¹ Among other, perhaps related goals is that of deriving the NM equation directly from the microscopic principles.

The well known analogy between the Quantum Hall Effects and High- T_c Superconductivity suggests that there should exist vortex lattices involving the so-called *filling factor* (microlocally a constant scalar) that plays a major role in the description of *Composite Particles*. The NM describe exactly this type of a vortex lattice. Simulation and theory show that this system conforms with the experimentally observed physical facts. It stretches the domain of applicability of the Maxwell theory to encompass phenomena such as the *Magnetic Oscillations*, *Magnetic Vortices*, *Charge Stripes* that occur in low-temperature electronic systems exposed to high magnetic fields.

There are other systems of PDE that admit vortex lattice solutions and are conceptually connected with Electromagnetism, like the well known Ginzburg–Landau equations valid within the framework of low T_c type-II superconductivity, or the Chern–Simons extension of these equations which, some researchers have suggested, may be more relevant to the Fractional Quantum Hall Effect and/or High- T_c Superconductivity, cf. [18]. The free variables of these equations are the so-called *order parameter* (a section of a complex line-bundle) and a $U(1)$ -principal connection, both of them containing topological information. In the case of NM, all the topological information is contained in one of the variables, i.e. the principal connection, while the other is a scalar function. An additional advantage of the NM is in that it remains meaningful in $3 + 1$ -dimensions just as well as in the two-dimensional setting. I would also like to mention that recently other researchers have introduced Nonlinear Maxwell Equations of another type in the context of the Quantum Hall Effects, cf. [3]. The NM theory presented in this and the preceding articles of mine is of a different nature. Finally, although this is far from my areas of expertise and the remark should be received as

¹ In fact, a few months after completion of this work I have discovered that the theory admits a suitable deformation that does not involve any additional fields, while possessing a Lagrangian formulation fit for quantization. At the same time, the phenomena discussed here remain valid within the framework of this extension. These new developments will be presented in forthcoming articles. I do not think this invalidates the possibility that there may be some other couplings as mentioned above.

completely ad hoc, I would also like to mention that yet another context in which foliations come in touch with the Quantum Hall Effect is that of noncommutative geometry, cf. [1].

Let me conclude with a question that may suggest yet another point of view. Namely, is there a coalescence between the nonlinear PDEs (in the form of the NM) and the (Quantum) Information Theory? As it was pointed out, construction of error correcting codes may unavoidably require manipulating quantum information at the topological level. Anyhow, this is how I have understood the essential thought in [2]. Adopting this paradigm would strongly suggest that the effective language of quantum computation should be constructed at many levels, including that of the mesoscopic field theory in parallel with the language derived from the basic principles as it is done now. Future research will likely better clarify these issues.

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